Contemporary Mathematics

The regular representations and the $A_n(V)$-algebras

Haisheng Li

Abstract. For a vertex operator algebra V, the regular representations are related to the $A_n(V)$-algebras and their bimodules, and induced V-modules from $A_n(V)$-modules are defined and studied in terms of the regular representations.

1. Introduction

In [Li2], for a vertex operator algebra V and a nonzero complex number z, a weak $V\otimes V$-module $\mathcal{D}_{P(z)}(V)$ was constructed out of the dual space V^*, and certain results of Peter-Weyl type were obtained. The weak $V\otimes V$-modules $\mathcal{D}_{P(z)}(V)$ were referred as regular representations. In [Li3], as a generalization, weak $V\otimes V$-modules $\mathcal{D}_{P(z)}(V,U)$ were constructed for any vector space U. Furthermore, Zhu's $A(V)$-theory ([Z1], [FZ]) was related to the regular representations in the spirit of the induced module theory for a Lie group (cf. [Ki]), and a notion of an induced V-module from an $A(V)$-module was formulated in terms of the regular representations. The induced V-module from an $A(V)$-module U was defined in [Li3] as follows: First consider linear functions from V to U, which are lifted from linear functions from $A(V)$ to U, or simply just linear functions from $A(V)$ to U. Second, it was shown that $\mathrm{Hom}(A(V),U)$ is a subspace of $\mathcal{D}_{P(-1)}(V,U)$, and what is more, $\mathrm{Hom}(A(V),U)$ and $\Omega(\mathcal{D}_{P(-1)}(V,U))$ ($\subset \mathrm{Hom}(V,U)$) coincide as natural $A(V)\otimes A(V)$-modules. Meanwhile, all the (left) $A(V)$-invariant functions from $A(V)$ to U give us the space $\mathrm{Hom}_{A(V)}(A(V),U)$, which is canonically isomorphic to U as an $A(V)$-module. Third, the induced module $\mathrm{Ind}_{A(V)}^{V}U$ was defined to be the submodule of $\mathcal{D}_{P(-1)}(V,U)$, generated by $\mathrm{Hom}_{A(V)}(A(V),U)$ ($=U$) under the action of $V\otimes\mathbb{C}$.

In [DLM2], as a generalization of Zhu's $A(V)$-theory, a family of associative algebras $A_n(V)$ were constructed and a family of functors Ω_n from the category of weak V-modules to the category of $A_n(V)$-modules and a family of functors M_n (with certain properties) from the category of $A_n(V)$-modules to the category of $\mathbb{N}$-graded weak V-modules were constructed. By definition, $\Omega_n(W)$ consists of each w such that $v_m w=0$ for homogeneous $v\in V$ and for $m\geq \mathrm{wt}v+n$. (Of course, $\Omega_n(W)$ can also be considered as the invariant space with respect to a certain Lie

1991 *Mathematics Subject Classification.* Primary 17B69; Secondary 17B68, 81R10.

This research was supported in part by NSF grants DMS-9616630 and DMS-9970496.

algebra.) In the case that W is a lowest weight generalized irreducible V-module, $\Omega_n(W)$ is the sum of the first n lowest weight subspaces.

In 1993, Zhu [Z2] gave a general construction of associative algebras from a vertex operator algebra for a certain purpose. The algebras $A_n(V)$ might be related to those algebras in a certain way.

In this paper, we shall relate $A_n(V)$-theory to the (generalized) regular representations of V on $\mathcal{D}_{P(-1)}(V,U)$. When $n\geq 1$, unlike the $n=0$ case [Li3], there are certain complicated factors. It is proved (Propositions 3.11, 3.15, and Corollary 3.17) that as vector spaces, $\mathrm{Hom}(A_n(V),U)$ is a subspace of $\Omega_n(\mathcal{D}_{P(-1)}(V,U))$. However, as natural $A_n(V)\otimes A_n(V)$-modules, $\mathrm{Hom}(A_n(V),U)$ is not a submodule. It turns out that the $A_n\otimes A_n(V)$-module structure on $\mathrm{Hom}(A_n(V),U)$ coincides with a twisted or deformed $A_n\otimes A_n(V)$-module structure on $\Omega_n(\mathcal{D}_{P(-1)}(V,U))$ with respect to a certain linear automorphism on $\Omega_n(\mathcal{D}_{P(-1)}(V,U))$ (Theorem 3.18). Using this connection we formulate a notion of induced V-module from an $A_n(V)$-module and we show that the induced modules are lowest weight generalized V-modules if the given $A_n(V)$-modules are irreducible.

An induced module theory from modules for a vertex operator subalgebra was established in [DLin]. As mentioned in [Li3], the notion of induced module defined here and the notion of induced module defined in [DLin] are different in nature.

This paper is organized as follows: In Section 2, we review the construction of the weak $V\otimes V$-module $\mathcal{D}_{P(z)}(W,U)$. In Section 3, we relate $A_n(V)$-modules $\mathrm{Hom}(A_n(W),U)$ with $\Omega_n(\mathcal{D}_{P(-1)}(W,U))$, and we define the induced V-module $\mathrm{Ind}_{A_n(V)}^{V}U$ for a given $A_n(V)$-module U.

2. The weak $V\otimes V$-module $\mathcal{D}_{P(z)}(W,U)$

In this section we shall recall from [Li3] the construction of the weak $V\otimes V$-module $\mathcal{D}_{P(z)}(W,U)$ and there are nothing new.

We use standard definitions and notations as given in [FLM] and [FHL]. A vertex operator algebra is denoted by $(V,Y,\mathbf{1},\omega)$, where $\mathbf{1}$ is the vacuum vector and ω is the Virasoro element, or simply by V. We also use the notion of weak module as defined in [DLM2]—A weak module satisfies all the axioms given in [FLM] and [FHL] for the notion of a module except that no grading is required.

We typically use letters $x,y,x_1,x_2,\dots$ for mutually commuting formal variables and $z,z_0,\dots$ for complex numbers. For a vector space U, $U[[x,x^{-1}]]$ is the vector space of all (doubly infinite) formal series with coefficients in U, $U((x))$ is the space of formal Laurent series in x, and $U((x^{-1}))$ is the space of formal Laurent series in x^{-1}. We emphasize the following standard formal variable convention:

$$(x_1-x_2)^n=\sum_{i\geq 0}(-1)^i\binom{n}{i}x_1^{n-i}x_2^i, \tag{2.1}$$

$$(x-z)^n=\sum_{i\geq 0}(-z)^i\binom{n}{i}x^{n-i}, \tag{2.2}$$

$$(z-x)^n=\sum_{i\geq 0}(-1)^i z^{n-i}\binom{n}{i}x^i \tag{2.3}$$

for $n\in\mathbb{Z},\ z\in\mathbb{C}^{\times}$.

For vector spaces U_1, U_2, a linear map $f \in \mathrm{Hom}(U_1, U_2)$ extends canonically to a linear map from $U_1[[x, x^{-1}]]$ to $U_2[[x, x^{-1}]]$. We shall use this canonical extension without any comments.

Let V be a vertex operator algebra. For $v \in V$, we set (cf. [FHL], [HL1])

$$Y^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \tag{2.4}$$

For a weak V-module W, because $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$ and $Y(u, x^{-1})w \in W((x^{-1}))$ for $u \in V,\ w \in W$, $Y^o(v, x)$ lies in $\mathrm{Hom}(W, W((x^{-1})))$. More generally, for any complex number z_0, $Y^o(v, x + z_0)$ lies in $\mathrm{Hom}(W, W((x^{-1})))$, where by definition

$$Y^o(v, x + z_0)w = (Y^o(v, y)w)|_{y=x+z_0} \tag{2.5}$$

for $w \in W$. Let W be a weak V-module and let U be a vector space, e.g., $U = \mathbb{C}$. For $v \in V,\ f \in \mathrm{Hom}(W, U)$, the compositions $fY^o(v, x)$ and $fY^o(v, x + z_0)$ for any complex number z_0 are elements of $(\mathrm{Hom}(W, U))[[x, x^{-1}]]$.

Let $\mathbb{C}(x)$ be the algebra of rational functions of x (and $\mathbb{C}[[x, x^{-1}]]$ be the vector space of all doubly infinite formal series in x with complex coefficients). The ι-maps $\iota_{x;0}$ and $\iota_{x;\infty}$ from $\mathbb{C}(x)$ to $\mathbb{C}[[x, x^{-1}]]$ are defined as follows: for any rational function $f(x)$, $\iota_{x;0}f(x)$ is the Laurent series expansion of $f(x)$ at $x = 0$ and $\iota_{x;\infty}f(x)$ is the Laurent series expansion of $f(x)$ at $x = \infty$. These are injective $\mathbb{C}[x, x^{-1}]$-linear maps. In terms of the formal variable convention, we have

$$\iota_{x;0}\left((x - z)^n f(x)\right) = (-z + x)^n \iota_{x;0}f(x), \tag{2.6}$$

$$\iota_{x;\infty}\left((x - z)^n f(x)\right) = (x - z)^n \iota_{x;\infty}f(x) \tag{2.7}$$

for $n \in \mathbb{Z},\ z \in \mathbb{C}^\times,\ f(x) \in \mathbb{C}(x)$.

Definition 2.1. Let W be a weak V-module, U a vector space and z a nonzero complex number. Define $\mathcal{D}_{P(z)}(W, U)$ to be the subspace of $\mathrm{Hom}(W, U)$, consisting of each f such that for $v \in V$, there exist $k, l \in \mathbb{N}$ such that

$$(x - z)^k x^l \langle u^*, fY^o(v, x)w\rangle \in \mathbb{C}[x] \tag{2.8}$$

for all $u^* \in U^*,\ w \in W$, or what is equivalent, for all $u^* \in U^*,\ w \in W$, the formal series

$$\langle u^*, fY^o(v, x)w\rangle,$$

an element of $\mathbb{C}((x^{-1}))$, absolutely converges in the domain $|x| > |z|$ to a rational function of the form $x^{-l}(x - z)^{-k}g(x)$ for $g(x) \in \mathbb{C}[x]$.

The following are equivalent definitions of $\mathcal{D}_{P(z)}(W, U)$ in terms of formal series:

Lemma 2.2. *Let $f \in \mathrm{Hom}(W, U)$. Then the following statements are equivalent:*

(a) $f \in \mathcal{D}_{P(z)}(W, U)$.

(b) For $v \in V$, there exist $k, l \in \mathbb{N}$ such that

$$(x - z)^k x^l fY^o(v, x) \in (\mathrm{Hom}(W, U))[[x]]. \tag{2.9}$$

(c) For $v \in V$, there exist $k, l \in \mathbb{N}$ such that for each $w \in W$,

$$(x - z)^k x^l fY^o(v, x)w \in U[x]. \tag{2.10}$$

Let $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U)$ and let $k,l\in \mathbb{N}$ be such that (2.10) holds. Then by changing variable we get

$$x^{k}(x+z)^{l}fY^{o}(v,x+z)w\in U[x] \tag{2.11}$$

for $w\in W$.

Definition 2.3. Let W,U and z be given as before. For

$$v\in V,\quad f\in \mathcal{D}_{P(z)}(W,U),$$

we define two elements $Y^{L}_{P(z)}(v,x)f$ and $Y^{R}_{P(z)}(v,x)f$ of $(\mathrm{Hom}(W,U))[[x,x^{-1}]]$ by

$$(Y^{L}_{P(z)}(v,x)f)(w) = (z+x)^{-l}\left((x+z)^{l}f(Y^{o}(v,x+z)w)\right) \tag{2.12}$$

$$(Y^{R}_{P(z)}(v,x)f)(w) = (-z+x)^{-k}\left((x-z)^{k}f(Y^{o}(v,x)w)\right) \tag{2.13}$$

for $w\in W$, where k,l are any pair of (possibly negative) integers such that (2.9) holds.

First, in view of (2.10) and (2.11), both $(z+x)^{-l}\left((x+z)^{l}f(Y^{o}(v,x+z)w)\right)$ and $(-z+x)^{-k}\left((x-z)^{k}f(Y^{o}(v,x)w)\right)$ lie in $U((x))$, so that $Y^{L}_{P(z)}(v,x)f$ and $Y^{R}_{P(z)}(v,x)f$ make sense. However, we are not allowed to remove the left-right brackets to cancel $(x-z)^{k}$ or $(x+z)^{l}$ because of the nonexistence of $(z+x)^{-l}f(Y^{o}(v,x+z)w)$ and $(-z+x)^{-k}f(Y^{o}(v,x)w)$. Second, they are also well defined, i.e., they are independent of the choice of the pair of integers k,l. Indeed, if k',l' are another pair of integers such that (2.9) holds, say for example, $k\ge k'$, then

$$\begin{aligned}
&(-z+x)^{-k}\left((x-z)^{k}fY^{o}(v,x)w\right)\\
=\;&(-z+x)^{-k}\left((x-z)^{k-k'}(x-z)^{k'}fY^{o}(v,x)w\right)\\
=\;&(-z+x)^{-k}(x-z)^{k-k'}\left((x-z)^{k'}fY^{o}(v,x)w\right)\\
=\;&(-z+x)^{-k'}\left((x-z)^{k'}fY^{o}(v,x)w\right).
\end{aligned} \tag{2.14}$$

From definition we immediately have:

Lemma 2.4. *For $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U)$,*

$$(z+x)^{l}Y^{L}_{P(z)}(v,x)f=(x+z)^{l}fY^{o}(v,x+z), \tag{2.15}$$

$$(-z+x)^{k}Y^{R}_{P(z)}(v,x)f=(x-z)^{k}fY^{o}(v,x), \tag{2.16}$$

where k,l are any pair of (maybe negative) integers such that (2.9) holds.

From the definition, $\langle u^{*},fY^{o}(v,x)w\rangle$ lies in the range of $\iota_{x;\infty}$ for $u^{*}\in U^{*},\ f\in \mathcal{D}_{P(z)}(W,U),\ v\in V,\ w\in W$. Then $\iota_{x;\infty}^{-1}\langle u^{*},fY^{o}(v,x)w\rangle$ is a well defined element of $\mathbb{C}(x)$. In terms of rational functions and the ι-maps we immediately have:

Lemma 2.5. *For $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U),\ u^{*}\in U^{*},\ w\in W$,*

$$\langle u^{*},(Y^{L}_{P(z)}(v,x)f)(w)\rangle=\iota_{x;0}\iota_{x;\infty}^{-1}\langle u^{*},fY^{o}(v,x+z)w\rangle, \tag{2.17}$$

$$\langle u^{*},(Y^{R}_{P(z)}(v,x)f)(w)\rangle=\iota_{x;0}\iota_{x;\infty}^{-1}\langle u^{*},fY^{o}(v,x)w\rangle. \tag{2.18}$$

THEOREM 2.6. *Let W be a weak V-module, U a vector space and z a nonzero complex number. Then the pairs $(\mathcal{D}_{P(z)}(W,U), Y^L_{P(z)})$ and $(\mathcal{D}_{P(z)}(W,U), Y^R_{P(z)})$ carry the structure of a weak V-module and the actions $Y^L_{P(z)}$ and $Y^R_{P(z)}$ of V on $\mathcal{D}_{P(z)}(W,U)$ commute. Furthermore, set*

$$Y_{P(z)} = Y^L_{P(z)} \otimes Y^R_{P(z)}. \tag{2.19}$$

Then the pair $(\mathcal{D}_{P(z)}(W,U), Y_{P(z)})$ carries the structure of a weak $V \otimes V$-module.

The following relation among $fY^o(v,x), Y^L(v,x)f$ and $Y^R(v,x)f$ holds [Li3]:

PROPOSITION 2.7. *Let $v \in V,\ f \in \mathcal{D}_{P(z)}(W,U)$. Then*

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x-z}{x_0}\right) fY^o(v,x) - x_0^{-1}\delta\left(\frac{z-x}{-x_0}\right) Y^R_{P(z)}(v,x)f \\ = \ & z^{-1}\delta\left(\frac{x-x_0}{z}\right) Y^L_{P(z)}(v,x_0)f. \end{aligned} \tag{2.20}$$

3. The associative algebras $A_n(V)$ and induced modules $\mathrm{Ind}_{A_n(V)}^V U$

In this section, the nonzero complex number z in the notion of weak $V \otimes V$-module $\mathcal{D}_{P(z)}(W,U)$ will be specified as -1. We shall simply use Y^L and Y^R for $Y^L_{P(-1)}$ and $Y^R_{P(-1)}$. Throughout this section, n will represent a nonnegative integer.

We shall need the following notions. A *generalized* V-module [HL1] is a weak V-module on which $L(0)$ semisimply acts. Then for a generalized V-module W we have the $L(0)$-eigenspace decomposition: $W = \coprod_{h \in \mathbb{C}} W_{(h)}$. Thus, a generalized V-module satisfies all the axioms defining the notion of a V-module ([FLM], [FHL]) except the two grading restrictions on homogeneous subspaces. If a generalized V-module furthermore satisfies the lower truncation condition (one of the two grading restrictions), it is called a *lower truncated* generalized module [H1].

A *lowest weight* generalized V-module is a generalized V-module such that $W = \coprod_{n \in \mathbb{N}} W_{(h+n)}$ for some $h \in \mathbb{C}$ and $W_{(h)}$ generates W as a weak V-module. Furthermore, if $W \neq 0$, we call the unique h *the lowest weight* of W.

Now we recall the construction of $A_n(V)$ algebra and some basic results from [DLM2].

DEFINITION 3.1. Let V be a vertex operator algebra and let $n \in \mathbb{N}$. Define a subspace $O_n(V)$ of V, linearly spanned by elements

$$(L(-1) + L(0))v, \tag{3.1}$$

$$\mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,u+n}}{x^{2n+2}} Y(u,x)v \tag{3.2}$$

for homogeneous $u, v \in V$. Define

$$\begin{aligned} u *_n v \ &= \ \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,u+n}}{x^{n+m+1}} Y(u,x)v \\ & \left(= \sum_{m=0}^{n} \binom{-n-1}{m} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,u+n}}{x^{n+m+1}} Y(u,x)v \right). \end{aligned} \tag{3.3}$$

We have:

LEMMA 3.2. [DLM2] *Let $u, v \in V$ be homogeneous. Then*

$$u *_n v - \sum_{m=0}^{n} \binom{-n-1}{m} (-1)^{n-m} \mathrm{Res}_x x^{-n-m-1} (1+x)^{\mathrm{wt} v+m-1} Y(v,x)u \in O_n(V).$$

Set $A_n(V) = V/O_n(V)$.

PROPOSITION 3.3. [DLM2] *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. Then*

(a) $O_n(V)$ is a two-sided ideal of the nonassociative algebra $(V, *_n)$ and the quotient algebra $A_n(V)$ is an associative algebra with $\mathbf{1} + O_n(V)$ as its identity element, with $\omega + O_n(V)$ being central and with an involution (anti-automorphism)

$$\theta : v + O_n(V) \mapsto e^{L(1)}(-1)^{L(0)} v + O_n(V). \tag{3.4}$$

(b) For each $n \geq 0$, the identity map of V gives rise to an algebra homomorphism ψ_n from $A_{n+1}(V)$ onto $A_n(V)$.

For any weak V-module W and any $n \in \mathbb{N}$, we define [DLM2]

$$\Omega_n(W) = \{w \in W \mid v_{\mathrm{wt} v+m} w = 0 \quad \text{for homogeneous } v \in V, \ m \geq n\}. \tag{3.5}$$

For $w \in \Omega_n(W)$ and for homogeneous $v \in V$, we have $x^{\mathrm{wt} v+n} Y(v,x)w \in W[[x]]$. When $n = 0$, we have $\Omega(W) = \Omega_0(W)$. Clearly,

$$\Omega_0(W) \subset \Omega_1(W) \subset \cdots. \tag{3.6}$$

We similarly define $\Omega_{-1}(W), \Omega_{-2}(W), \ldots$. Since $\mathrm{wt}\mathbf{1} = 0$ and $\mathbf{1}_r w = \delta_{r,-1} w$, we have $\Omega_{-n}(W) = 0$ for $n \geq 1$.

The following result was proved in [DLM2]:

PROPOSITION 3.4. *Let W be a weak V-module and let $n \geq 0$. Then $\Omega_n(W)$ is an $A_n(V)$-module where $v + O_n(V)$ acts as $v_{\mathrm{wt} v-1}$ for homogeneous $v \in V$.*

Let W_1, W_2 be weak V-modules and let ψ be a V-homomorphism from W_1 to W_2. It is clear that $\psi(\Omega_n(W_1)) \subset \Omega_n(W_2)$ and the restriction $\Omega_n(\psi) := \psi|_{\Omega_n(W_1)}$ is an $A_n(V)$-homomorphism. It is routine to check that we have obtained a functor Ω_n from the category of weak V-modules to the category of $A_n(V)$-modules.

LEMMA 3.5. *Let W be a weak V-module and set*

$$\mathcal{S}(W) = \cup_{n \geq 0} \Omega_n(W). \tag{3.7}$$

Let $u \in V$ be homogeneous and let $r \in \mathbb{Z}$. Then

$$u_r \Omega_n(W) \subset \Omega_n(W) \tag{3.8}$$

if $r \geq \mathrm{wt} u - 1$, and

$$u_r \Omega_n(W) \subset \Omega_{n+\mathrm{wt} u-r-1}(W) \tag{3.9}$$

if $r < \mathrm{wt} u - 1$. In particular, $\mathcal{S}(W)$ is a sub-weak-module of W. Furthermore,

$$\Omega_n(\mathcal{S}(W)) = \Omega_n(W). \tag{3.10}$$

PROOF. Let $w \in \Omega_n(W)$, let $v \in V$ be homogeneous and let $m \in \mathbb{Z}$. By Borcherds commutator formula,

$$
\begin{aligned}
(3.11)\qquad & v_{\mathrm{wt}v+m}u_r w \\
&= u_r v_{\mathrm{wt}v+m}w + \sum_{i\geq 0}\binom{\mathrm{wt}v+m}{i}(v_i u)_{\mathrm{wt}v+m+r-i}w \\
&= u_r v_{\mathrm{wt}v+m}w + \sum_{i\geq 0}\binom{\mathrm{wt}v+m}{i}(v_i u)_{\mathrm{wt}(v_i u)+m+r-\mathrm{wt}u+1}w.
\end{aligned}
$$

Then the first part follows immediately. Since $\mathcal{S}(W)$ is a submodule of W, we have $\Omega_n(\mathcal{S}(W)) \subset \Omega_n(W)$. It is easy to see that $\Omega_n(W) \subset \Omega_n(\mathcal{S}(W))$. This completes the proof. □

We shall need the result that $L(1)$ is locally nilpotent on $\mathcal{S}(W)$ for any weak V-module W. To prove this result, we recall from [Li3] the following result, which is a reformulation of a result in [DLM2] (Remark 3.3):

LEMMA 3.6. *Let W be a weak V-module, $w \in W$. Let $u, v \in V$ and let $k \in \mathbb{Z}$ be such that*

$$x^k Y(u,x)w \in W[[x]], \tag{3.12}$$

or equivalently,

$$u_{k+m}w = 0 \quad \text{for } m \geq 0. \tag{3.13}$$

Then for $p, q \in \mathbb{Z}$,

$$u_p v_q w = \sum_{i=0}^{s}\sum_{j\geq 0}\binom{p-k}{i}\binom{k}{j}(u_{p-k-i+j}v)_{q+k+i-j}w. \tag{3.14}$$

where s is any nonnegative integer such that $x^{s+1+q}Y(v,x)w \in W[[x]]$.

As an immediate consequence we have ([DM] and [Li1]):

COROLLARY 3.7. *Let W be a weak V-module and let $w \in W$. Set*

$$\langle w\rangle = \text{linear span } \{v_m w \mid v \in V,\ m \in \mathbb{Z}\}. \tag{3.15}$$

Then $\langle w\rangle$ is the sub-weak-module of W, generated by w.

LEMMA 3.8. *Let W be a weak V-module. Then for any r homogeneous vectors $v^1, \dots, v^r \in V$,*

$$v^1_{m_1}\cdots v^r_{m_r}\Omega_n(W) = 0 \tag{3.16}$$

for $m_i \in \mathbb{Z}$ with

$$m_1 + \cdots + m_r \geq \mathrm{wt}v^1 + \cdots + \mathrm{wt}v^r - r + n.$$

In particular, for homogeneous $v \in V$ and for $m \geq \mathrm{wt}v$,

$$(v_m)^n\Omega_n(W) = 0. \tag{3.17}$$

PROOF. We shall prove the first part by induction on r. From the definition of $\Omega_n(W)$, the lemma is true for $r = 1$. Assume it is true for any r homogeneous vectors in V. Now let $v^1, \dots, v^r, v^{r+1} \in V$ be homogeneous and let $m_i \in \mathbb{Z}$ with

$$m_1 + \cdots + m_r + m_{r+1} \geq \mathrm{wt}v^1 + \cdots + \mathrm{wt}v^{r+1} - (r+1) + n. \tag{3.18}$$

Set

$$u = v^r, \;\; v = v^{r+1}, \;\; p = m_r, \;\; q = m_{r+1}.$$

Since $w \in \Omega_n(W)$, in Lemma 3.6, we may take $k = \mathrm{wt}u + n = \mathrm{wt}v^r + n$. Let s be any nonnegative integer such that $x^{s+1+q}Y(v,x)w \in W[[x]]$. By Lemma 3.6, we have

$$\begin{aligned} &u_p v_q w \\ = &\sum_{i=0}^{s}\sum_{j\geq 0}\binom{p-\mathrm{wt}u-n}{i}\binom{\mathrm{wt}u+n}{j}(u_{p-\mathrm{wt}u-n-i+j}v)_{q+\mathrm{wt}u+n+i-j}w. \end{aligned} \tag{3.19}$$

Notice that

$$\begin{aligned} \mathrm{wt}(u_{p-\mathrm{wt}u-n-i+j}v) &= \mathrm{wt}u+\mathrm{wt}v+\mathrm{wt}u+n+i-j-1-p \\ &= 2\mathrm{wt}v^r+\mathrm{wt}v^{r+1}+n+i-j-1-p. \end{aligned} \tag{3.20}$$

Thus

$$\begin{aligned} &m_1+\cdots+m_{r-1}+(q+\mathrm{wt}u+n+i-j) \\ \geq\; &\mathrm{wt}v^1+\cdots+\mathrm{wt}v^{r+1}-(r+1)+n+(q+\mathrm{wt}u+n+i-j)-m_r-m_{r+1} \\ =\; &\mathrm{wt}v^1+\cdots+\mathrm{wt}v^{r-1}+\mathrm{wt}(u_{p-\mathrm{wt}u-n-i+j}v)-r+n. \end{aligned}$$

Then it follows from the inductive hypothesis that

$$\begin{aligned} &v^1_{m_1}\cdots v^{k+1}_{m_{k+1}}w \\ = &\sum_{i=0}^{s}\sum_{j\geq 0}\binom{p-k}{i}\binom{k}{j}v^1_{m_1}\cdots v^{k-1}_{m_{k-1}}(u_{p-k-i+j}v)_{q+k+i-j}w \\ = &\;0. \end{aligned} \tag{3.21}$$

This finishes the induction and concludes the proof. □

In view of Lemma 3.8, noticing that $L(1) = \omega_2$ and $\mathrm{wt}\omega = 2$, we immediately have:

Corollary 3.9. *Let W be a weak V-module, let $v \in V$ be homogeneous and let $m \geq \mathrm{wt}v$. Then v_m is locally nilpotent on $\mathcal{S}(W)$. In particular, $L(1)$ is locally nilpotent on $\mathcal{S}(W)$.*

Let W be a weak V-module. We define $O'_n(W)$ to be the subspace of W, linearly spanned by elements of the form:

$$v \circ_n w := \mathrm{Res}_x x^{-2n-2}(1+x)^{\mathrm{wt}v+n}Y(v,x)w \tag{3.22}$$

for $w \in W$ and for homogeneous $v \in V$. The proof of Lemma 2.1.2 of [Z1] with minor necessary changes directly gives:

Lemma 3.10. *Let W be a weak V-module, let $w \in W$, and let $v \in V$ be homogeneous. Then*

$$\mathrm{Res}_x x^{-2n-2-r}(1+x)^{\mathrm{wt}v+n+s}Y(v,x)w \in O'_n(W) \tag{3.23}$$

for $r \geq s \geq 0$.

In the following there will be several module structures on a certain vector space. For this reason, we shall use $\Omega_n(W, Y_W)$ including the vertex operator map Y_W in the notation for $\Omega_n(W)$. Since $\mathrm{Hom}(-,U)$ is a contravariant functor for the category of vector spaces, for any vector spaces A, B and any surjective linear map

$g \in \mathrm{Hom}(A, B)$, we have an injective linear map $\mathrm{Hom}(g, U)$ from $\mathrm{Hom}(B, U)$ into $\mathrm{Hom}(A, U)$. In particular, if B is a quotient space of A, we may naturally identify $\mathrm{Hom}(B, U)$ as a subspace of $\mathrm{Hom}(A, U)$.

PROPOSITION 3.11. *Let W be a weak V-module and let U be a vector space. Set $A'_n(W) = W/O'_n(W)$. Then*

$$\mathrm{Hom}(A'_n(W), U) = \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^L) \cap \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^R). \tag{3.24}$$

Furthermore, elements α of $\mathrm{Hom}(A'_n(W), U)$, a natural subspace of $\mathrm{Hom}(W, U)$, are characterized by the following property:

$$x^{\mathrm{wt}v+n}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(v, x) \in (\mathrm{Hom}(W, U))[[x]] \tag{3.25}$$

for homogeneous $v \in V$.

PROOF. Let T be the set defined by the property (3.25). We shall prove

$$\begin{aligned}&\mathrm{Hom}(A'_n(W), U) \subset T \subset \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^L) \cap \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^R) \subset \\ &\subset T \subset \mathrm{Hom}(A'_n(W), U).\end{aligned}$$

Let $\alpha \in \mathrm{Hom}(A'_n(W), U)$ and let $v \in V$ be homogeneous. Then for any $m \geq 0$,

$$\begin{aligned} & \mathrm{Res}_x x^{\mathrm{wt}v+n+m}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(v, x)w \\ = \ & \mathrm{Res}_x x^{\mathrm{wt}v+n+m}(x+1)^{\mathrm{wt}v+n}\alpha Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \\ = \ & (-1)^{\mathrm{wt}v}\mathrm{Res}_x x^{2n+m}(1+x^{-1})^{\mathrm{wt}v+n}\alpha Y(e^{xL(1)}v, x^{-1})w \\ = \ & (-1)^{\mathrm{wt}v}\mathrm{Res}_x x^{-2n-m-2}(1+x)^{\mathrm{wt}v+n}\alpha Y(e^{x^{-1}L(1)}v, x)w \\ = \ & 0 \end{aligned} \tag{3.26}$$

because (Lemma 3.10)

$$\begin{aligned} & \mathrm{Res}_x x^{-2n-m-2}(1+x)^{\mathrm{wt}v+n}Y(e^{x^{-1}L(1)}v, x)w \\ = \ & \sum_{i\geq 0}\frac{1}{i!}\mathrm{Res}_x x^{-2n-m-2-i}(1+x)^{\mathrm{wt}(L(1)^i v)+n+i}Y(L(1)^i v, x)w \\ \in \ & O'_n(W). \end{aligned} \tag{3.27}$$

This proves (3.25). Since $Y^o(v, x)w \in W((x^{-1}))$ for $w \in W$, (3.25) implies

$$x^{\mathrm{wt}v+n}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(v, x)w \in U[x]. \tag{3.28}$$

By changing variable we get

$$(x-1)^{\mathrm{wt}v+n}x^{\mathrm{wt}v+n}\alpha Y^o(v, x-1) \in (\mathrm{Hom}(W, U))[[x]]. \tag{3.29}$$

By Lemma 2.2, $\alpha \in \mathcal{D}_{P(-1)}(W, U)$ and by Lemma 2.4

$$x^{\mathrm{wt}v+n}(1+x)^{\mathrm{wt}v+n}Y^R(v, x)\alpha = x^{\mathrm{wt}v+n}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(v, x), \tag{3.30}$$

$$(-1+x)^{\mathrm{wt}v+n}x^{\mathrm{wt}v+n}Y^L(v, x)\alpha = (x-1)^{\mathrm{wt}v+n}x^{\mathrm{wt}v+n}\alpha Y^o(v, x-1). \tag{3.31}$$

Consequently,

$$x^{\mathrm{wt}v+n}Y^R(v, x)\alpha, \ \ x^{\mathrm{wt}v+n}Y^L(v, x)\alpha \in (\mathrm{Hom}(W, U))[[x]]. \tag{3.32}$$

That is, $\alpha \in \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^L) \cap \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^R)$.

Conversely, let $\alpha \in \Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R)$. Then (3.32) holds for each homogeneous $v \in V$. Recall (2.20) with $f=\alpha,\ z=-1$:

$$\begin{aligned}(3.33)\qquad & x_0^{-1}\delta\left(\frac{x+1}{x_0}\right)\alpha Y^o(v,x) - x_0^{-1}\delta\left(\frac{1+x}{x_0}\right)Y^R_{P(z)}(v,x)\alpha \\ =\ & -\delta(-x+x_0)Y^L_{P(z)}(v,x_0)\alpha.\end{aligned}$$

Applying $\mathrm{Res}_{x_0}x^{\mathrm{wt}v+n}x_0^{\mathrm{wt}v+n}$ to (3.33), then using (3.32) and the fundamental properties of delta function we get

$$\begin{aligned}(3.34)\qquad & x^{\mathrm{wt}v+n}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(v,x)\\ =\ & x^{\mathrm{wt}v+n}(1+x)^{\mathrm{wt}v+n}Y^R(v,x)\alpha\\ & -\mathrm{Res}_{x_0}x^{\mathrm{wt}v+n}x_0^{\mathrm{wt}v+n}\delta(-x+x_0)Y^L(v,x_0)\alpha\\ =\ & x^{\mathrm{wt}v+n}(1+x)^{\mathrm{wt}v+n}Y^R(v,x)\alpha\\ \in\ & (\mathrm{Hom}(W,U))[[x]].\end{aligned}$$

Furthermore, for any $w \in W$,

$$\begin{aligned}& \mathrm{Res}_x x^{-2n-2}(1+x)^{\mathrm{wt}v+n}\alpha Y(v,x)w\\ =\ & \mathrm{Res}_x x^{-2n-2}(1+x)^{\mathrm{wt}v+n}\alpha Y^o(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1})w\\ =\ & \mathrm{Res}_x x^{2n}(1+x^{-1})^{\mathrm{wt}v+n}\alpha Y^o(e^{x^{-1}L(1)}(-x^2)^{L(0)}v,x)w\\ =\ & \mathrm{Res}_x(-1)^{\mathrm{wt}v}x^{\mathrm{wt}v+n}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(e^{x^{-1}L(1)}v,x)w\\ =\ & \sum_{i\ge 0}(-1)^{\mathrm{wt}v}\frac{1}{i!}\mathrm{Res}_x x^{\mathrm{wt}v-i+n}(x+1)^{\mathrm{wt}v+n}\alpha Y^o(L(1)^iv,x)w\\ =\ & \sum_{i\ge 0}(-1)^{\mathrm{wt}v}\frac{1}{i!}\mathrm{Res}_x x^{\mathrm{wt}(L(1)^iv)+n}(x+1)^{\mathrm{wt}(L(1)^iv)+n+i}\alpha Y^o(L(1)^iv,x)w\\ =\ & 0.\end{aligned}$$

Thus $\alpha(O_n'(W))=0$, hence $\alpha \in \mathrm{Hom}(A_n'(W),U)$. This completes the proof. $\square$

It follows from Theorem 2.6 and Proposition 3.4 that $\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R)$ is a natural $A_n(V)$-module. Since Y^L and Y^R commute, $\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R)$ is also a weak V-module under the vertex operator map Y^L. Then it follows from Proposition 3.4 again that

$$\Omega_n\left(\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R),Y^L\right)$$

is an $A_n(V)\otimes A_n(V)$-module. Clearly,

$$\begin{aligned}(3.35)\qquad & \Omega_n\left(\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R),Y^L\right)\\ =\ & \Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R).\end{aligned}$$

Thus, $\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R)$ is an $A_n(V)\otimes A_n(V)$-module. For convenience, we refer to this $A_n(V)\otimes A_n(V)$-module structure as the *canonical module structure*. From definition, we have

$$(3.36)\quad \Omega_n(\mathcal{D}_{P(-1)}(W,U))\subseteq \Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R).$$

(The equality of (3.36) holds when $n=0$, but the equality does not hold for $n\ge 1$.) It is easy to see that $\Omega_n(\mathcal{D}_{P(-1)}(W,U))$ is an $A_n(V)\otimes A_n(V)$-submodule.

Motivated by [Li3] for $n=0$, we should identify $\mathrm{Hom}(A_n'(W),U)$ with

$$\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R)$$

as natural $A_n(V)\otimes A_n(V)$-modules. We shall prove that $A'_n(W)$ just like $A_n(V)$ has a natural $A_n(V)\otimes A_n(V)$-module structure and so does $\mathrm{Hom}(A'_n(W),U)$. It turns out that the $A_n(V)\otimes A_n(V)$-module $\mathrm{Hom}(A'_n(W),U)$ is naturally isomorphic to

$$\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R)$$

with a deformed $A_n(V)\otimes A_n(V)$-module structure.

To achieve our goal we shall need the following result (cf. [Li2], Remark 2.10):

PROPOSITION 3.12. *Let (E,Y_E) be a weak V-module on which $L(1)$ is locally nilpotent, and let z_0 be any complex number. For $v\in V$, we define*

$$Y_E^{[z_0]}(v,x)=Y_E(e^{-z_0(1+z_0x)L(1)}(1+z_0x)^{-2L(0)}v,x/(1+z_0x)). \tag{3.37}$$

Then the pair $(E,Y_E^{[z_0]})$ carries the structure of a weak V-module and $e^{-z_0L(1)}$ is a V-isomorphism from (E,Y_E) to $(E,Y_E^{[z_0]})$. Furthermore, for homogeneous $v\in V$ and for $m\in\mathbb{Z}$, we have

$$\begin{aligned}&\mathrm{Res}_x x^m Y_E^{[z_0]}(v,x)\\ =\ &\mathrm{Res}_x x^m(1-z_0x)^{2\mathrm{wt}v-m-2}Y_E(e^{-z_0(1-z_0x)^{-1}L(1)}v,x).\end{aligned} \tag{3.38}$$

In particular,

$$\begin{aligned}&\mathrm{Res}_x x^{\mathrm{wt}v-1}Y^{[z_0]}(v,x)w\\ =\ &\mathrm{Res}_x x^{\mathrm{wt}v-1}(1-z_0x)^{\mathrm{wt}v-1}Y\left(e^{-z_0(1-z_0x)^{-1}L(1)}v,x\right).\end{aligned} \tag{3.39}$$

PROOF. Recall the conjugation formula (5.2.38) of [FHL]:

$$\begin{aligned}&e^{-x_1L(1)}Y(v,x)e^{x_1L(1)}\\ =\ &Y(e^{-x_1(1+x_1x)L(1)}(1+x_1x)^{-2L(0)}v,x/(1+x_1x)).\end{aligned} \tag{3.40}$$

Because $L(1)$ is locally nilpotent on E, we may set $x_1=z_0$, so that we have

$$\begin{aligned}&e^{-z_0L(1)}Y_E(v,x)e^{z_0L(1)}\\ =\ &Y_E(e^{-z_0(1+z_0x)L(1)}(1+z_0x)^{-2L(0)}v,x/(1+z_0x))\\ =\ &Y_E^{[z_0]}(v,x).\end{aligned} \tag{3.41}$$

Then the first part of the proposition follows immediately.

By changing variable $x=y/(1-z_0y)$ we get

$$\begin{aligned}&\mathrm{Res}_x x^m Y_E^{[z_0]}(v,x)\\ =\ &\mathrm{Res}_x x^m Y_E(e^{-z_0(1+z_0x)L(1)}(1+z_0x)^{-2L(0)}v,x/(1+z_0x))\\ =\ &\mathrm{Res}_y y^m(1-z_0y)^{-m-2}Y_E(e^{-z_0(1+z_0y)^{-1}L(1)}(1-z_0y)^{2L(0)}v,y)\\ =\ &\mathrm{Res}_y y^m(1-z_0y)^{2\mathrm{wt}v-m-2}Y_E(e^{-z_0(1-z_0y)^{-1}L(1)}v,y).\end{aligned}$$

This completes the proof. □

By definition we have

$$\begin{aligned}&(Y^{[z_0]})^{[-z_0]}(v,x)\\ =\ &Y^{[z_0]}(e^{z_0(1-z_0x)L(1)}(1-z_0x)^{-2L(0)}v,x/(1-z_0x))\\ =\ &Y(e^{-z_0(1+z_0x)L(1)}(1+z_0x)^{-2L(0)}e^{z_0(1-z_0x)L(1)}(1-z_0x)^{-2L(0)}v,x).\end{aligned} \tag{3.42}$$

Recall (5.3.3) of [FHL]:

$$x_1^{-L(0)}L(1)x_1^{L(0)} = x_1L(1). \tag{3.43}$$

From this we immediately get

$$x_1^{-L(0)}e^{xL(1)}x_1^{L(0)} = e^{xx_1L(1)}. \tag{3.44}$$

In view of (3.44) we have

$$e^{-z_0(1+z_0x)L(1)}(1+z_0x)^{-2L(0)}e^{z_0(1-z_0x)L(1)}(1-z_0x)^{-2L(0)} = 1, \tag{3.45}$$

hence

$$(Y^{[z_0]})^{[-z_0]}(v,x) = Y(v,x). \tag{3.46}$$

Continuing with Proposition 3.12 we have:

PROPOSITION 3.13. *Let (E, Y_E) be a weak V-module on which $L(1)$ is locally nilpotent and let z_0 be any complex number. Then*

$$\Omega_n(E, Y_E) = \Omega_n(E, Y_E^{[z_0]}). \tag{3.47}$$

Furthermore, $e^{-z_0L(1)}$ is an $A_n(V)$-isomorphism from $\Omega_n(E, Y)$ to $\Omega_n(E, Y^{[z_0]})$.

PROOF. From (3.38) we easily get

$$\Omega_n(E, Y_E) \subset \Omega_n(E, Y_E^{[z_0]}). \tag{3.48}$$

Using this and the fact that $Y_E = (Y_E^{[z_0]})^{[-z_0]}$, we get

$$\Omega_n(E, Y_E^{[z_0]}) \subset \Omega_n(E, Y_E). \tag{3.49}$$

This proves (3.47). The second part follows from Proposition 3.12 immediately. $\square$

We shall use Proposition 3.13 for $z_0 = 0, -1, 1$. Let W and U be given as before. Set

$$E = \mathcal{S}(\mathcal{D}_{P(-1)}(W, U)). \tag{3.50}$$

In view of Lemma 3.5 we have

$$\Omega_n(\mathcal{D}_{P(-1)}(W, U)) = \Omega_n(E) \tag{3.51}$$

and it follows from Corollary 3.9 that $L(1)$ is locally nilpotent on E, so that we can apply Propositions 3.12 and 3.13 to E.

Let W be a weak V-module. For homogeneous $v \in V$ and for $w \in W$, we define

$$\begin{aligned} & v *_n w \\ = \quad & \sum_{m=0}^{n} \binom{-n-1}{m} \mathrm{Res}_x x^{-n-m-1}(1+x)^{\mathrm{wt}\, v+n} Y(v,x)w, \end{aligned} \tag{3.52}$$

$$\begin{aligned} & w *_n v \\ = \quad & \sum_{m=0}^{n} \binom{-n-1}{m} (-1)^{n-m} \mathrm{Res}_x x^{-n-m-1}(1+x)^{\mathrm{wt}\, v+m-1} Y(v,x)w. \end{aligned} \tag{3.53}$$

Then extend the definition by linearity.

Now, we are in a position to prove our key result.

PROPOSITION 3.14. *Let W be a weak V-module and let U be a vector space. Let*

$$f \in \mathrm{Hom}(A'_n(W), U) = \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^L) \cap \Omega_n(\mathcal{D}_{P(-1)}(W, U), Y^R)$$

and $w \in W$. Then

$$\left(\mathrm{Res}_x x^{\mathrm{wt}\, v} (Y^L)^{[1]}(v, x) f\right)(w) = f(w *_n v) \tag{3.54}$$

$$\left(\mathrm{Res}_x x^{\mathrm{wt}\, v} (Y^R)^{[-1]}(v, x) f\right)(w) = f(\theta(v) *_n w) \tag{3.55}$$

for homogeneous $v \in V$, where

$$\theta(v) = e^{L(1)}(-1)^{L(0)} v \tag{3.56}$$

(cf. (3.4)).

PROOF. First, using (3.44) we get ([FHL], (5.3.1)):

$$e^{xL(1)}(-x^{-2})^{L(0)} e^{x^{-1}L(1)} = (-x^2)^{-L(0)}, \tag{3.57}$$

$$e^{xL(1)}(-x^{-2})^{L(0)} e^{(x+1)^{-1}L(1)} = e^{x/(x+1)L(1)}(-x^{-2})^{L(0)}. \tag{3.58}$$

Because

$$\left(\sum_{m=0}^{n} \binom{-n-1}{m} (-1)^{n+1-m} x^m\right)(-1+x)^{n+1} \in 1 + x^{n+1}\mathbb{C}[[x]], \tag{3.59}$$

for $k \geq n$,

$$\mathrm{Res}_x x^{\mathrm{wt}\, v + k} Y^L(v, x) f = 0. \tag{3.60}$$

Since for any homogeneous $u \in V$,

$$(-1+x)^{\mathrm{wt}\, u + n} Y^L(u, x) f = (x-1)^{\mathrm{wt}\, u + n} f Y^o(u, x-1) \tag{3.61}$$

(Proposition 3.11), we have

$$\begin{aligned} &(-1+x)^{\mathrm{wt}\, v + n} Y^L(e^{(-1+x)^{-1}L(1)} v, x) f \\ = \;&(x-1)^{\mathrm{wt}\, v + n} f Y^o(e^{(x-1)^{-1}L(1)} v, x-1), \end{aligned} \tag{3.62}$$

noting that $\mathrm{wt}L(1)^i v = \mathrm{wt}v - i$ for $i \geq 0$. Using (3.39) and all the above information we have

$$
\begin{aligned}
& \left(\mathrm{Res}_x x^{\mathrm{wt}v-1}(Y^L)^{[1]}(v,x)f\right)(w) \\
= \; & \mathrm{Res}_x(-1)^{\mathrm{wt}v-1}x^{\mathrm{wt}v-1}(-1+x)^{\mathrm{wt}v-1}\left(Y^L(e^{(-1+x)^{-1}L(1)}v,x)f\right)(w) \\
= \; & \mathrm{Res}_x \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{\mathrm{wt}v+n-m}x^{m+\mathrm{wt}v-1}(-1+x)^{\mathrm{wt}v+n} \cdot \\
& \cdot \left(Y^L(e^{(-1+x)^{-1}L(1)}v,x)f\right)(w) \\
= \; & \mathrm{Res}_x \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{\mathrm{wt}v+n-m}x^{m+\mathrm{wt}v-1}(x-1)^{\mathrm{wt}v+n} \cdot \\
& \cdot f\left(Y^o(e^{(x-1)^{-1}L(1)}v,x-1)w\right) \\
= \; & \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{\mathrm{wt}v+n-m} \\
& \cdot \mathrm{Res}_x(x+1)^{m+\mathrm{wt}v-1}x^{\mathrm{wt}v+n}f\left(Y^o(e^{x^{-1}L(1)}v,x)w\right) \\
= \; & \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{\mathrm{wt}v+n-m} \cdot \\
& \cdot \mathrm{Res}_x(x+1)^{m+\mathrm{wt}v-1}x^{\mathrm{wt}v+n}f(Y(e^{xL(1)}(-x^{-2})^{L(0)}e^{x^{-1}L(1)}v,x^{-1})w) \\
= \; & \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{\mathrm{wt}v+n-m} \cdot \\
& \cdot \mathrm{Res}_x(x+1)^{m+\mathrm{wt}v-1}x^{\mathrm{wt}v+n}f(Y((-x^2)^{-L(0)}v,x^{-1})w) \\
= \; & \mathrm{Res}_x \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{n-m}(x+1)^{m+\mathrm{wt}v-1}x^{-\mathrm{wt}v+n}f(Y(v,x^{-1})w) \\
= \; & \mathrm{Res}_x \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{n-m}(x^{-1}+1)^{m+\mathrm{wt}v-1}x^{\mathrm{wt}v-n-2}f(Y(v,x)w) \\
= \; & \mathrm{Res}_x \sum_{m=0}^{n} \binom{-n-1}{m}(-1)^{n-m}x^{-n-m-1}(1+x)^{\mathrm{wt}v+m-1}f(Y(v,x)w) \\
= \; & f(w *_n v).
\end{aligned}
$$

Similarly, using the fact

$$
\left(\sum_{m=0}^{n} \binom{-n-1}{m}x^m\right)(1+x)^{\mathrm{wt}v+n} \in 1 + x^{n+1}\mathbb{C}[[x]] \tag{3.63}
$$

we have

$$
\begin{aligned}
&\left(\mathrm{Res}_x x^{\mathrm{wt}\, v-1}(Y^R)^{[-1]}(v,x)f\right)(w)\\
=\;& \mathrm{Res}_x x^{\mathrm{wt}\, v-1}(1+x)^{\mathrm{wt}\, v-1}\left(Y^R(e^{(1+x)^{-1}L(1)}v,x)f\right)(w)\\
=\;& \mathrm{Res}_x \sum_{m=0}^{n}\binom{-n-1}{m}x^{m+\mathrm{wt}\, v-1}(1+x)^{\mathrm{wt}\, v+n}\left(Y^R(e^{(1+x)^{-1}L(1)}v,x)f\right)(w)\\
=\;& \mathrm{Res}_x \sum_{m=0}^{n}\binom{-n-1}{m}x^{m+\mathrm{wt}\, v-1}(x+1)^{\mathrm{wt}\, v+n}f\left(Y^o(e^{(x+1)^{-1}L(1)}v,x)w\right)\\
=\;& \mathrm{Res}_x \sum_{m=0}^{n}\binom{-n-1}{m}x^{m+\mathrm{wt}\, v-1}(x+1)^{\mathrm{wt}\, v+n}\cdot\\
&\cdot f\left(Y(e^{xL(1)}(-x^{-2})^{L(0)}e^{(x+1)^{-1}L(1)}v,x^{-1})w\right)\\
=\;& \mathrm{Res}_x \sum_{m=0}^{n}\binom{-n-1}{m}(-1)^{\mathrm{wt}\, v}x^{m-\mathrm{wt}\, v-1}(x+1)^{\mathrm{wt}\, v+n}\cdot\\
&\cdot f\left(Y(e^{x/(x+1)L(1)}(-x^{-2})^{L(0)}v,x^{-1})w\right)\\
=\;& \mathrm{Res}_x \sum_{m=0}^{n}\binom{-n-1}{m}(-1)^{\mathrm{wt}\, v}x^{-m+\mathrm{wt}\, v-1}(x^{-1}+1)^{\mathrm{wt}\, v+n}\cdot\\
&\cdot f\left(Y(e^{(1+x)^{-1}L(1)}(-x^{2})^{L(0)}v,x)w\right)\\
=\;& \mathrm{Res}_x \sum_{m=0}^{n}\binom{-n-1}{m}(-1)^{\mathrm{wt}\, v}x^{-n-m-1}(1+x)^{\mathrm{wt}\, v+n}f\left(Y(e^{(1+x)^{-1}L(1)}v,x)w\right)\\
=\;& \sum_{m=0}^{n}\sum_{i\geq 0}\frac{1}{i!}\binom{-n-1}{m}(-1)^{\mathrm{wt}\, v}\cdot\\
&\cdot \mathrm{Res}_x x^{-n-m-1}(1+x)^{\mathrm{wt}(L(1)^i v)+n}f\left(Y(L(1)^i v,x)w\right)\\
=\;& f(\theta(v)*_n w).
\end{aligned}
$$

This completes the proof. □

One can in principle use similar arguments to those in [Z1], [FZ] and [DLM2] to show that the left and right actions of V on W, defined by (3.52) and (3.53), give rise to an $A_n(V)$-bimodule structure on $A_n'(W)$, or $A_n(W)$ defined below. (From the proof of Theorem 2.3 of [DLM2], to prove the associativity for the right action it seems that we need to prove at least one more combinatorial identity in addition to those proved in [DLM2].) As a matter of fact, this easily follows from Proposition 3.14 and the (canonical and deformed) $A_n(V)\otimes A_n(V)$-module structures on

$$\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,U),Y^R).$$

PROPOSITION 3.15. *Let W be a weak V-module. Then the left and right actions of V on W, defined by (3.52) and (3.53), give rise to an $A_n(V)$-bimodule structure on $A_n'(W)$.*

Proof. Let $U=\mathbb{C}$. For homogeneous $v\in V$ we set

$$o_L^{[1]}(v)=\mathrm{Res}_x x^{\mathrm{wt} v-1}(Y^L)^{[1]}(v,x), \tag{3.64}$$

$$o_R^{[-1]}(v)=\mathrm{Res}_x x^{\mathrm{wt} v-1}(Y^R)^{[1]}(v,x). \tag{3.65}$$

Then extend the definition by linearity. It follows from Theorem 2.6 and Propositions 3.4, 3.12 and 3.13 that $o_L^{[1]}\otimes o_R^{[-1]}$ gives rise to an $A_n(V)\otimes A_n(V)$-structure on

$$\Omega_n(\mathcal{D}_{P(-1)}(W,\mathbb{C}),Y^L)\cap\Omega_n(\mathcal{D}_{P(-1)}(W,\mathbb{C}),Y^R)=\mathrm{Hom}(A_n'(W),\mathbb{C}).$$

In particular,

$$o_L^{[1]}(O_n(V))=o_R^{[-1]}(O_n(V))=0. \tag{3.66}$$

The following arguments are classical and routine in nature. Let $u,v\in V$ be homogeneous and let $w\in W$. For any $f\in \mathrm{Hom}(A_n'(W),\mathbb{C})$, because

$$o_L^{[1]}(v)f\in \mathrm{Hom}(A_n'(W),\mathbb{C}),$$

in view of Proposition 3.14 we have

$$f((u\circ_n w)*_n v)=\langle o_L^{[1]}(v)f,u\circ_n w\rangle=0. \tag{3.67}$$

Since f is arbitrary, we must have

$$(u\circ_n w)*_n v\in O_n'(W). \tag{3.68}$$

Using Proposition 3.14 and (3.66) we have

$$\langle f,w*_n(u\circ_n v)\rangle=\langle o_L^{[1]}(u\circ_n v)f,w\rangle=0 \tag{3.69}$$

for every $f\in\mathrm{Hom}(A_n'(W),\mathbb{C})$. Consequently,

$$w*_n(u\circ_n v)\in O_n'(W). \tag{3.70}$$

Similarly, using the fact that θ gives rise to the involution θ of $A_n(V)$ (Proposition 3.3) we have

$$v*_n(u\circ_n w),\quad (u\circ_n v)*_n w\in O_n'(W). \tag{3.71}$$

Then the left action and right action of V on W give rise to a left action and right action of $A_n(V)$ on $A_n'(W)$. The rest can be proved similarly. □

Motivated by the definition of $O_n(V)$ we define

$$O_n(W)=O_n'(W)+(L(-1)+L(0))W. \tag{3.72}$$

The proof of of Lemma 2.1 of [DLM2] directly gives:

Lemma 3.16. *Let W be a weak V-module, let $w\in W$ and let $v\in V$ be homogeneous. Then*

$$v*_n w-w*_n v\equiv \mathrm{Res}_x(1+x)^{\mathrm{wt} v-1}Y(v,x)w \mod O_n(W). \tag{3.73}$$

Set

$$A_n(W)=W/O_n(W). \tag{3.74}$$

Then we have:

Corollary 3.17. *The subspace $O_n(W)$ of W is stable under the left and right actions of V on W, defined by (3.52) and (3.53), and the quotient space $A_n(W)$ is an $A_n(V)$-bimodule which is the quotient module of $A_n'(W)$ modulo $O_n(W)/O_n'(W)$.*

Proof. In view of Proposition 3.15 we only need to prove

$$(L(-1)w + L(0)w) *_n v \in O_n(W), \tag{3.75}$$
$$v *_n (L(-1)w + L(0)w) \in O_n(W) \tag{3.76}$$

for $v \in V$, $w \in W$. Let us assume v is homogeneous. First, from the proof of Lemma 2.2 in [DLM2] we have

$$(L(-1)v + L(0)v) *_n w = (-1)^n(2n+1)\binom{2n+1}{n}(v \circ_n w) \in O_n(W). \tag{3.77}$$

Then using the fact

$$\begin{aligned} [L(-1)+L(0), Y(v,x)] &= (1+x)Y(L(-1)v,x) + Y(L(0)v,x) \\ &= (1+x)(d/dx)Y(v,x) + (\mathrm{wt}v)Y(v,x), \end{aligned} \tag{3.78}$$

we get

$$\begin{aligned} & v *_n (L(-1)+L(0))w \\ = \; & (L(-1)+L(0))(v *_n w) + (L(-1)v + L(0)v) *_n w \in O_n(W). \end{aligned} \tag{3.79}$$

Using Lemma 3.16 and (3.78), we get

$$\begin{aligned} & (L(-1)w + L(0)w) *_n v \\ \equiv \; & v *_n (L(-1)w + L(0)w) \\ & -\mathrm{Res}_x(1+x)^{\mathrm{wt}v-1}Y(v,x)(L(-1)w + L(0)w) \mod O_n(W) \\ \equiv \; & -\mathrm{Res}_x(1+x)^{\mathrm{wt}v-1}Y(v,x)(L(-1)w + L(0)w) \mod O_n(W) \\ = \; & -\mathrm{Res}_x(1+x)^{\mathrm{wt}v-1}(L(-1)+L(0))Y(v,x)w \\ & +\mathrm{Res}_x(1+x)^{\mathrm{wt}v}(d/dx)Y(v,x)w + \mathrm{Res}_x(\mathrm{wt}v)(1+x)^{\mathrm{wt}v-1}Y(v,x)w \\ = \; & -\mathrm{Res}_x(1+x)^{\mathrm{wt}v-1}(L(-1)+L(0))Y(v,x)w \\ \equiv \; & 0 \mod O_n(W). \end{aligned} \tag{3.80}$$

(This argument is also similar to one in the proof Lemma 2.2 of [DLM2].) □

Because $A'_n(W)$ is an $A_n(V)$-bimodule and θ is an involution of $A_n(V)$, from the classical fact $\mathrm{Hom}(A'_n(W), U)$ becomes an $A_n(V) \otimes A_n(V)$-module with

$$((a_1,a_2)f)(w) = f(\theta(a_2)wa_1) \tag{3.81}$$

for $a_1, a_2 \in A_n(V)$, $f \in \mathrm{Hom}(A'_n(W), U)$, $w \in A_n(V)$. We refer to this $A_n(V) \otimes A_n(V)$-module structure as the *canonical dual module structure.*

Combining Propositions 3.14 with 3.13 we immediately have:

Theorem 3.18. *Let W be a weak V-module and U a vector space. Let*

$$\eta : \mathrm{Hom}(A'_n(W), U) \to \Omega_n(\mathcal{D}_{P(-1)}(W,U), Y^L) \cap \Omega_n(\mathcal{D}_{P(-1)}(W,U), Y^R)$$

be the natural identification map (Proposition 3.11). Then the linear map

$$\sigma := e^{L^R(1) - L^L(1)} \circ \eta$$

is an $A_n(V) \otimes A_n(V)$-isomorphism where $\mathrm{Hom}(A'_n(W), U)$ is equipped with the canonical dual $A_n(V) \otimes A_n(V)$-module structure and

$$\Omega_n(\mathcal{D}_{P(-1)}(W,U), Y^L) \cap \Omega_n(\mathcal{D}_{P(-1)}(W,U), Y^R)$$

is equipped with the canonical module structure.

Let U be an $A_n(V)$-module. Then $U = \mathrm{Hom}_{A_n(V)}(A_n(V), U)$. We also have the following A_n-module inclusion relations:

$$\mathrm{Hom}_{A_n(V)}(A_n(V), U) \subset \mathrm{Hom}_{A_n(V)}(A'_n(V), U) \subset \mathrm{Hom}_{\mathbb{C}}(A'_n(V), U).$$

With Theorem 3.18 we may and we should identify U as a submodule of the $A_n(V)$-module $\Omega_n(\mathcal{D}_{P(-1)}(V, U), Y^L)$.

DEFINITION 3.19. *Let U be an $A_n(V)$-module. We define $\mathrm{Ind}_{A_n(V)}^V U$ to be the submodule of $(\mathcal{D}_{P(-1)}(V, U), Y^L)$, generated by U $(= \mathrm{Hom}_{A_n(V)}(A_n(V), U))$.*

Using the proof of Lemma 3.14 in [Li3] with some minor changes, we have:

PROPOSITION 3.20. *Let W be a weak V-module and let U be an irreducible $A_n(V)$-submodule of $\Omega_n(W)$. Then the weak submodule M of W generated by U is a lowest weight generalized V-module such that $M_{(h)} = U$ for some $h \in \mathbb{C}$ and $M_{(k+h)} = 0$ for $k < -n$. In particular, if U is an irreducible $A_n(V)$-module, then $\mathrm{Ind}_{A_n(V)}^V U$ is a lowest weight generalized V-module with U being the homogeneous subspace of some weight h such that the lowest weight is no smaller than $h - n$.*

References

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068-3071.

[DL] C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Math., **Vol.** 112, Birkhäuser, Boston, 1993.

[DLin] C. Dong and Z. Lin, Induced modules for vertex operator algebras, *Commun. Math. Phys.* **179** (1996), 157-184.

[DLM1] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Annalen* **310** (1998), 571-700.

[DLM2] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Alg.* **206** (1998), 67-96.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Memoirs Amer. Math. Soc. **104**, 1993.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., **Vol. 134**, Academic Press, Boston, 1988.

[FZ] I. Frenkel and Y.-C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.

[H1] Y.-Z. Huang, A theory of tensor products for module category of a vertex operator algebra, IV, *J. Pure Appl. Alg.* **100** (1995), 173-216.

[HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module category of a vertex operator algebra, I, *Selecta Mathematica* **1** (1995), 699-756.

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module category of a vertex operator algebra, II, *Selecta Mathematica* **1** (1995), 757-786.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module of a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995), 141-171.

[Ki] A. A. Kirillov, *Elements of the Theory of Representations*, GMT 220, Springer-Verlag Berlin Heidelberg New York, 1976.

[Li1] H.-S. Li, Representation theory and tensor product theory for vertex operator algebras, Ph.D. thesis, Rutgers University, 1994.

[Li2] H.-S. Li, Regular representations of vertex operator algebras, I, math-qa/9908108.

[Li3] H.-S. Li, The regular representation, Zhu's $A(V)$-theory and induced modules, math-qa/9909007.

[Z1] Y.-C. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-302; Vertex operator algebras, elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.

[Z2] Y.-C. Zhu, Talk at the Conformal Field Theory Seminar of mathematics department, Rutgers, 1993.

Department of Mathematical Sciences, Rutgers University-Camden, Camden, NJ 08102

E-mail address: hli@crab.rutgers.edu